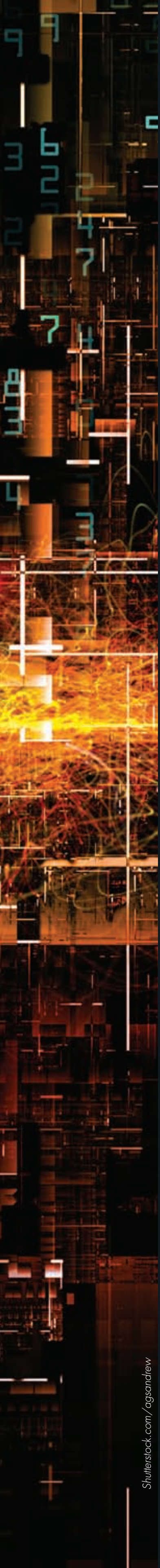

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8

TERMINOLOGY

addition of matrices
column matrix
determinant
dimension
element
identity matrix
inverse
invertible
leading diagonal
matrix (matrices)
multiplication of matrices
non-singular
order
row matrix
scalar
singular
square matrix
zero matrix

Prior learning

MATRICES

MATRIX ARITHMETIC

8.01 Matrices
8.02 Scalar multiplication of matrices
8.03 Addition and subtraction of matrices
8.04 Matrix multiplication
8.05 Identities and inverses
8.06 Matrix equations
Chapter summary
Chapter review

MATRIX ARITHMETIC

- understand the matrix definition and notation (ACMSM051)
- define and use addition and subtraction of matrices, scalar multiplication, matrix multiplication, multiplicative identity and inverse (ACMSM052)
- calculate the determinant and inverse of 2×2 matrices and solve matrix equations of the form $AX = B$, where A is a 2×2 matrix and X and B are column vectors. (ACMSM053) AC

8.01 MATRICES

Matrices are one of the most important structures in mathematics. There is evidence of the study of matrices in the form of 'magic squares' in Chinese literature dating back to 650 BCE. The earliest mathematical application of matrices appears to be in solving linear equations, where they proved to be particularly useful in solving simultaneous equations. The study of matrices was developed by Arab mathematicians and spread to India when the Arabs conquered parts of the Indian subcontinent in the 7th century. The study of matrices was then applied to additional branches of mathematics as well as astronomy. The term 'matrix' first appeared in texts in 1848, after which time a number of famous mathematicians worked on matrix theory.

Matrices can be used to encrypt numerical data and they have a wide range of applications in computer graphics. The power of matrices lies in the fact that they provide a concise way to write and process information that would otherwise be unmanageable due to their volume and complexity.

IMPORTANT

A **matrix** (the plural is **matrices**) is a rectangular array of numbers enclosed in large parentheses () or brackets [ ]. In typing, a capital letter in boldface is used as the symbol for a matrix. In handwriting, you can underline the capital letter with a wavy line.

For example, $\mathbf{B} = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & 0 & 5 & 6 \end{bmatrix}$ could be written as $\underset{\sim}{B}$.

Each number in the matrix is called an **element** of the matrix.

The **size** (**dimension** or **order**) of a matrix is always given with the *rows first*.

Matrix **B** above is a 2×4 matrix.

A great deal of information is shown as tables of numbers. For some tables, there will be blank cells where it does not make sense to have an entry, but in most cases every cell will have a number, even if it is zero.

For example, the following table shows the wins, draws and losses in the 2011–2012 Tri-Series Cricket held in Australia between Australia, Sri Lanka and India.

Team	Wins	Draws	Losses	Points
Sri Lanka	4	1	3	19
Australia	4	0	4	19
India	3	1	4	15

The match data is shown by the numbers in the rows and columns. You can see that there are 3 rows and 4 columns of data; they form a rectangular array of numbers.

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IMPORTANT

Elements of a matrix are referred to using the lower case letter of the matrix name and subscripts to indicate the row and column of the element. A general matrix may be represented by

$$\underset{\text{j changes} \rightarrow}{\overset{n \text{ columns}}{m \text{ rows} \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & c_{22} & c_{23} & \cdots \\ c_{31} & c_{32} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \searrow \end{bmatrix} \; i \text{ changes} \downarrow}}$$

This means that matrix **C** has elements c_{ij} and is of size $m \times n$.

Example 1

a Write the matrix **C** for the cricket results shown in the table at the top of the page.
b What is the value of c_{23}?
c What is the size of the matrix?

Solution

a The headings and team names are omitted. There are 3 rows and 4 columns of data.

$$\begin{bmatrix} 4 & 1 & 3 & 19 \\ 4 & 0 & 4 & 19 \\ 3 & 1 & 4 & 15 \end{bmatrix}$$

b c_{23} is the value in the 2nd row and 3rd column. $c_{23} = 4$

c Size = no. of rows × no. of columns. The size of the matrix is 3×4

If a matrix **A** has the same number of rows as columns, then we say that it is a **square matrix**.

e.g. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

In a square matrix the elements a_{ii}, with $i = 1, 2, 3, \ldots$, are called **diagonal elements**.

A **diagonal matrix** is a square matrix with all the non-diagonal elements zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

A matrix with one row is called a **row matrix**.

e.g. $\begin{bmatrix} 2 & 1 & -3 & 0 \end{bmatrix}$

A matrix with one column is called a **column matrix**.

e.g. $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

EXERCISE 8.01 Matrices

Concepts and techniques

1 Example 1 Which of the following represent matrices?

a $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

b $\begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix}$

c $\begin{matrix} 1 & 4 \\ 3 & 2 \end{matrix}$

d $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & -1 \\ & -3 & -1 \end{bmatrix}$

e $\begin{bmatrix} \sqrt{3} & \frac{5}{6} \\ -6 & 2 \end{bmatrix}$

f $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 $\mathbf{X} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -5 & 2 & -2 \\ -2 & 3 & 1 & -2 \end{bmatrix}$. If possible, write down each of the following.

a x_{12} b x_{21} c x_{34} d x_{22} e x_{32}

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3 $\mathbf{R} = \begin{bmatrix} 0 & 4 & 7 & 2 & 3 \\ -1 & 3 & 5 & 0 & 8 \\ 8 & 6 & 1 & -2 & 10 \\ 3 & -9 & 0 & 3 & 4 \end{bmatrix}$

Express the relationship between each pair of elements below in simple forms like $2r_{41} = r_{32}$.

a r_{31} and r_{25}
b r_{14} and r_{45}
c r_{32} and r_{15}
d r_{42} and r_{44}
e r_{25} and r_{34}
f r_{35} and r_{34}

4 Name each of the following matrices stating their size, type and any other distinguishing features.

a $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
b $\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$
c $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
d $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
e $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

Reasoning and communication

5 A manufacturer builds desktop computers with a variety of central processor speeds and hard drive capacities. The result of a stock inventory is shown in the table below.

Hard Drive	1.86 GHz	2.0 GHz	3.8 GHz	2.33 GHz	2.67 GHz	2.8 GHz	3.0 GHz	3.2 GHz	3.5 GHz
160 GB	3	2	4	1	5	0	3	6	1
250 GB	1	8	1	7	1	8	0	5	8
320 GB	4	0	5	6	4	5	9	1	7
500 GB	7	9	2	0	1	1	10	9	1

a Write the information in the table as a 4 × 9 matrix **M**.
b Write the value of m_{32}.
c Write the value of m_{21}.
d Write the value of m_{27}.
e What is the order of **M**?

6 On a particular trading day, the exchange rates for Australian dollars (AUD), United States dollars (USD), Great Britain pounds (GBP) and Japanese yen (JPY) were as follows.

1 AUD = 1.05664 USD, 0.65720 GBP, 94.3892 JPY
1 USD = 0.94642 AUD, 0.62192 GBP, 89.3362 JPY
1 GBP = 1.52148 AUD, 1.60769 USD, 143.625 JPY
1 JPY = 0.01060 AUD, 0.01120 USD, 0.00696 GBP

a What is the equivalence of 1 AUD in USD?
b Put the information in a 4 × 4 matrix **E** in the order AUD, USD, GPB and JPY.
c What is the value of *e* As e_{24}?
d What is the value of e_{32}?
e In this matrix, what is the relationship between e_{ij} and e_{ji}, where $i \neq j$?

7 The following table indicates extreme temperatures and rainfall levels for each state in Australia as at 1 January 2013.

Capital	Max	Year	Min	Year	Rainfall	Year
South Australia	50.7	1960	−8.2	1976	1852.6	1917
Western Australian	50.5	1998	−7.2	2008	2380.6	2000
New South Wales	49.7	1939	−23.0	1994	4539.7	1950
Queensland	49.5	1972	−10.6	1965	12461.0	2000
Victoria	48.8	2009	−11.7	1970	3738.5	1956
Northern Territory	48.3	1960	−7.5	1976	2953.2	2000
Tasmania	42.2	2009	−13.0	1983	4504.1	1948

Bureau of Meteorology

a What would be the size of the matrix **W** formed by this information?

b What is the value of w_{11}?

c What is the value of w_{76}?

d What is the maximum value for $(c_{i1} - c_{i3})$, and what does this represent?

8 An ecologist interested in the predation of species in a freshwater pool studied the diets of water beetles, mosquito larvae, fingerlings and adult fish. She found that mosquito larvae and fingerlings both ate microscopic algae and microscopic animals in the water, but that fingerlings also ate approximately 3 mosquito larvae each day. Water beetles ate about 5 mosquito larvae and 2 fingerlings each day, and adult fish ate about 6 fingerlings and 4 water beetles each day.

a Write the information as the 4×4 matrix **P**, showing the predation of species on each other, with the animals in the order: mosquito larvae, fingerlings, water beetles and adult fish.

b What is the significance of the row of zeros for mosquito larvae?

c What is the meaning of p_{33}?

9 Construct the matrix (d_{ij}) if $1 \le i \le 3$, $1 \le j \le 4$, $d_{ij} = 1$ if $i = j$, $d_{ij} = 2$ if $i > j$ and $d_{ij} = 0$ if $i < j$.

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8.02 SCALAR MULTIPLICATION OF MATRICES

Inflation has the effect of increasing prices over time. The retail prices of certain 'milk' varieties were recorded in a particular year and are shown in the table below.

Milk type	1 L bottle	2 L bottle
Whole milk	$1.85	$3.09
Low fat	$1.94	$3.46
High calcium	$2.04	$3.95
Soy milk	$2.35	$4.55

In matrix form this is: $\mathbf{M} = \begin{bmatrix} 1.85 & 3.09 \\ 1.94 & 3.46 \\ 2.04 & 3.95 \\ 2.35 & 4.55 \end{bmatrix}$

Inflation is currently at approximately 3% per annum, meaning that prices of products such as milk can expect to increase by 3% each year.

So to increase each element in the matrix **M** by 3%, we obtain a new matrix **N** by multiplying by 1.03 as follows:

$$\mathbf{N} = 1.03 \times \begin{bmatrix} 1.85 & 3.09 \\ 1.94 & 3.46 \\ 2.04 & 3.95 \\ 2.35 & 4.55 \end{bmatrix} = \begin{bmatrix} 1.03\times1.85 & 1.03\times3.09 \\ 1.03\times1.94 & 1.03\times3.46 \\ 1.03\times2.04 & 1.03\times3.95 \\ 1.03\times2.35 & 1.03\times4.55 \end{bmatrix} \approx \begin{bmatrix} 1.91 & 3.18 \\ 2.00 & 3.56 \\ 2.10 & 4.07 \\ 2.42 & 4.69 \end{bmatrix}$$

Multiplication of a matrix by a number is called multiplication by a **scalar**.

IMPORTANT

Given a matrix **A** and a constant c, the matrix $c\mathbf{A}$ is defined by $c\mathbf{A} = (ca_{ij})$.

The new matrix is referred to as a **scalar multiple** of the original matrix.

Example 2

Given that

$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 & -3 & -5 \\ -2 & 5 & 4 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix}$, find

a $3\mathbf{A}$ b $-\mathbf{B}$ c $\frac{1}{2}\mathbf{C}$

Solution

a Multiply every element by 3.

$$3\mathbf{A} = 3\begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 9 \\ -3 & 3 \\ 9 & 0 \end{bmatrix}$$

b Multiply every element by −1.

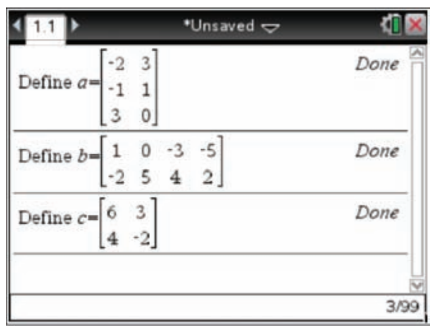
$$-1\mathbf{B} = -1 \times \begin{bmatrix} 1 & 0 & -3 & -5 \\ -2 & 5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 & 5 \\ 2 & -5 & -4 & -2 \end{bmatrix}$$

c Multiply every element by $\frac{1}{2}$.

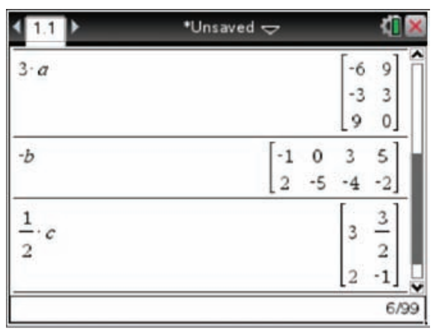
$$\frac{1}{2}\mathbf{C} = \frac{1}{2} \times \begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1\frac{1}{2} \\ 2 & -1 \end{bmatrix}$$

TI-Nspire CAS

From the menu, choose 1: Actions 1: Define. Press A and use menu, 7:Matrix & Vector, 1:Create and 1: Matrix. Select 3 rows and 2 columns and fill to make **A**.

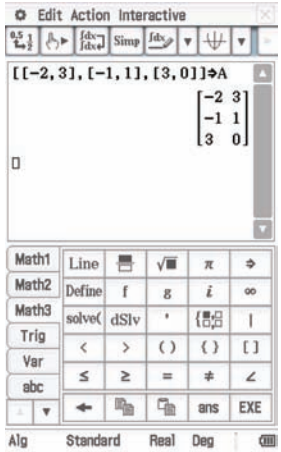


When you've made all three, find 3**A**, −**B** and use the template key and choose the fraction template for $\frac{1}{2}$**C**.



ClassPad

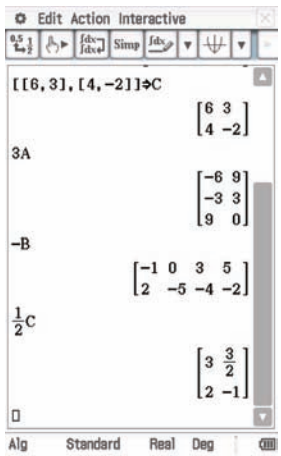
Use the $\sqrt[\text{Main}]{\alpha}$ application. .
Start with square brackets obtainable from the keyboard shown after you press [Keyboard] then tap [Math3].
Enter each row in square brackets, with the numbers separated by commas.
Finish with [⇒] A.



Follow a similar process to enter **B** and **C**.
To multiply by 3, type 3**A** then tap [EXE] or press [EXE].

Repeat for $-\mathbf{B}$ and $\frac{1}{2}\mathbf{C}$

For $-\mathbf{B}$, use the [(−)] key.



EXERCISE 8.02 Scalar multiplication of matrices

Concepts and techniques

1 Example 2 Given that $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$, find:

a $3\mathbf{A}$ b $0.4\mathbf{B}$ c $5\mathbf{B}$ d $-4\mathbf{A}$ e $\mathbf{B} \div 3$

2 Given that $\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix}$, find:

a $-2\mathbf{X}$ b $0.5\mathbf{Y}$ c $-\mathbf{Y}$ d $3\mathbf{X}$

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3 Given that $\mathbf{M} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix}$, find:

a $5\mathbf{M}$
b $-3\mathbf{N}$
c $4\mathbf{M}$
d $-5\mathbf{N}$

Reasoning and communication

4 The wholesale prices ($) of some paint products are shown in the following matrix. The first column represents the prices of 100 mL tubes, the second the prices of 250 mL tubes and the third the prices of 400 mL tubes. The first row is for *black*, the second for *red*, the third for *yellow* and the fourth for *white*.

$$\mathbf{W} = \begin{bmatrix} 2.40 & 4.90 & 8.40 \\ 4.20 & 7.90 & 11.80 \\ 3.80 & 8.40 & 11.00 \\ 5.90 & 11.80 & 14.40 \end{bmatrix}$$

a The retailer applies a mark-up of 70%. Calculate the matrix for the retail prices.
b A GST of 10% has to be added to get the final price for the customer. What is the matrix for the prices actually paid by the customer?

5 A local cattle farmer sells the following head of cattle at the local saleyard each Monday for 5 weeks. Matrix **Y** represents the number of yearlings sold and matrix **C** represents the number of calves sold. The rows represent the different Mondays that he went to the saleyards.

$$\mathbf{Y} = \begin{bmatrix} 20 \\ 16 \\ 25 \\ 23 \\ 19 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 4 \\ 1 \\ 7 \\ 2 \\ 3 \end{bmatrix}$$

a If yearlings sold each week for $350, write the matrix to show the income for the five weeks from the sale of the yearlings.
b If calves sold for $60 each week, write a matrix to show the income for the five weeks from the sale of the calves.
c What was the total income from all sales over the five weeks?

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8.03 ADDITION AND SUBTRACTION OF MATRICES

The sales of mobile telephone plans for January for three Telstra outlets are shown below.

Telstra outlet	Cap plan	Phone plan	Member plan
Queen Street Mall, Brisbane	150	28	35
Castletown Shopping Centre, Townsville	89	15	22
Pacific Fair Shopping Centre, Gold Coast	125	12	43

Telstra website

This information can be represented as the matrix $\mathbf{J} = \begin{bmatrix} 150 & 28 & 35 \\ 89 & 15 & 22 \\ 125 & 12 & 43 \end{bmatrix}$.

If the sales figures for February are represented as $\mathbf{F} = \begin{bmatrix} 182 & 33 & 47 \\ 112 & 23 & 17 \\ 138 & 36 & 46 \end{bmatrix}$, how can **J** and **F** be combined to show the sales figures for the various locations for the January/February period?

The January sales for the Queen Street Mall outlet are shown in the first row of **J** and the February sales for the same store are shown in the first row of **F**. This means that it only makes sense to add corresponding rows of **J** and **F** to find the totals. As the first columns of **J** and **F** show the number of cap plans sold in each month, it makes sense to add corresponding elements within each row.

In doing this, the total sales by Telstra (**T**) can be shown as follows.

$$\mathbf{T} = \begin{bmatrix} 150 & 28 & 35 \\ 89 & 15 & 22 \\ 125 & 12 & 43 \end{bmatrix} + \begin{bmatrix} 182 & 33 & 47 \\ 112 & 23 & 17 \\ 138 & 36 & 46 \end{bmatrix} = \begin{bmatrix} 150+182 & 28+33 & 35+47 \\ 89+112 & 15+23 & 22+17 \\ 125+138 & 12+36 & 43+46 \end{bmatrix} = \begin{bmatrix} 332 & 61 & 82 \\ 201 & 38 & 39 \\ 263 & 48 & 89 \end{bmatrix}$$

IMPORTANT

The process of adding matrices is known as **matrix addition**.

Given two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, we define the sum $\mathbf{A} + \mathbf{B}$ as the $m \times n$ matrix $\mathbf{C}$ such that $c_{ij} = a_{ij} + b_{ij}$ for all i, j.

Corresponding elements are added. That is, $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.

Example 3

Find $\mathbf{A} + \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

Solution

Add corresponding elements, i.e., $a_{11} + b_{11}$, $a_{12} + b_{12}$, etc

$$\begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+6 & 1+5 & 2+4 \\ 9+3 & 8+4 & 7+5 \end{bmatrix}$$

Write $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \end{bmatrix}$$

If two matrices are not the same size, they cannot be added.

Work out $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -3 \\ -2 & 5 & 4 \end{bmatrix}$. What do you find?

Work out $\begin{bmatrix} -3 & -4 & 3 \\ 2 & -5 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -3 \\ -2 & 5 & 4 \end{bmatrix}$. What do you find?

IMPORTANT

The identity for matrix addition is the **zero matrix** $\mathbf{O} = (o_{ij})$, where $o_{ij} = 0$ for all i, j.

If required, the size of the zero matrix is included as a subscript, as in $\mathbf{O}_{4\times 5}$.

The inverse for matrix addition is the matrix such that every element is the opposite sign of the given matrix. This is actually $(-1)\mathbf{A}$, normally written as $-\mathbf{A}$.

The subtraction of matrices is considered as addition of the **additive inverse**.

$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = \mathbf{A} + (-\mathbf{B})$

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Example 4

Given $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & -3 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 6 & -4 \\ 2 & -3 \\ 3 & 2 \end{bmatrix}$, find:

a $\mathbf{A} - \mathbf{C}$

b $\mathbf{B} - \mathbf{A} - \mathbf{C}$

c $2\mathbf{A} - \mathbf{C}$

Solution

a Treat $\mathbf{A} - \mathbf{C}$ as $\mathbf{A} + (-\mathbf{C})$

$$\mathbf{A} - \mathbf{C} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 \\ -3 & 4 \\ 0 & -2 \end{bmatrix}$$

b Treat $\mathbf{B} - \mathbf{A} - \mathbf{C}$ as $\mathbf{B} + (-\mathbf{A}) + (-\mathbf{C})$

$$\mathbf{B} - \mathbf{A} - \mathbf{C} = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 1 & -1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -1 & 0 \\ -5 & -5 \end{bmatrix}$$

c Treat $2\mathbf{A} - \mathbf{C}$ as $2\mathbf{A} + (-\mathbf{C})$

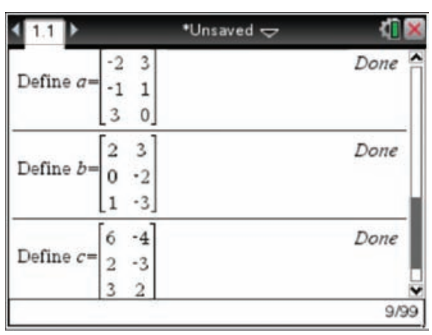
$$2\mathbf{A} - \mathbf{C} = 2 \times \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 6 \\ -2 & 2 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

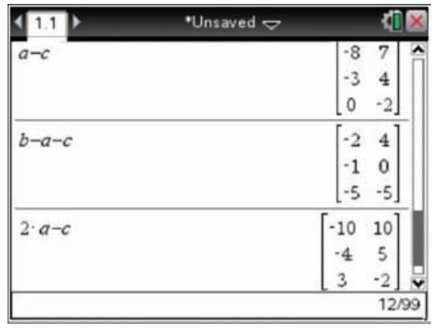
$$= \begin{bmatrix} -10 & 10 \\ -4 & 5 \\ 3 & -2 \end{bmatrix}$$

TI-Nspire CAS

Define each matrix (see page 288).

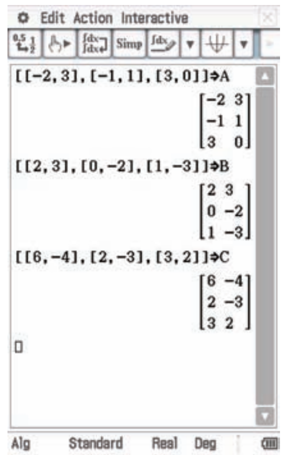


Now calculate **A** – **C**, **B** – **A** – **C** and 2**A** – **C**.

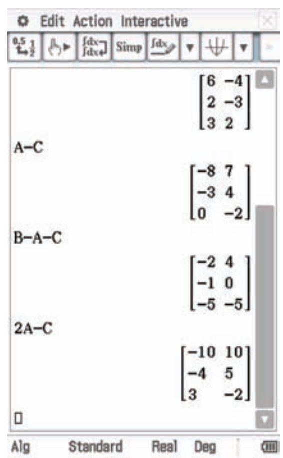


ClassPad

Define **A**, **B** and **C** as before.



Enter each matrix operation, followed by EXE.



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IMPORTANT

Laws for addition and scalar multiplication of matrices

The following laws hold for any matrices **A**, **B** and **C** and scalars r and s.

1. Matrix addition is associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
2. Matrix addition is commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
3. The additive identity **O** has all elements equal to zero and is called the **zero matrix**.
4. The additive inverse of $\mathbf{A} = (a_{ij})$ is $-\mathbf{A} = (-a_{ij})$.
5. Cancellation laws hold for scalar multiplication of a matrix:
   $$r\mathbf{A} = r\mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{B}\ (r \neq 0)$$
   $$r\mathbf{A} = s\mathbf{A} \Leftrightarrow r = s\ (\mathbf{A} \neq \mathbf{O})$$
6. Distributive laws hold for:
   Scalar multiplication over addition: $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
   Scalar multiplication over matrix addition: $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
7. An associative law holds for multiplication and scalar multiplication:
   $$(rs)\mathbf{A} = r(s\mathbf{A})$$

EXERCISE 8.03 Addition and subtraction of matrices

WS Addition and subtraction of matrices

Concepts and techniques

1 Example 3 Given that

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & -10 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}, \text{ find:}$$

a $\mathbf{C} + \mathbf{A}$ b $\mathbf{B} + \mathbf{D}$ c $\mathbf{C} + \mathbf{B}$ d $5\mathbf{A} + \mathbf{B}$ e $3\mathbf{B} + 2\mathbf{D}$

2 Example 4 Given that

$$\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix} \text{ and } \mathbf{Z} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 2 & 4 \\ 0 & 6 & 2 \\ 7 & 0 & 0 \end{bmatrix}, \text{ find:}$$

a $\mathbf{Z} + \mathbf{X}$ b $\mathbf{X} + \mathbf{Z}$ c $\mathbf{Y} + \mathbf{Z}$
d $4\mathbf{Z} + \mathbf{X}$ e $2\mathbf{Z} + 3\mathbf{X} + 2\mathbf{Y}$

3 Given that

$$\mathbf{L} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 6 & -5 & 4 & -3 \\ -1 & 8 & -9 & 2 \\ 4 & 7 & -2 & 5 \end{bmatrix}, \text{ find:}$$

a $\mathbf{L} + \mathbf{M}$ b $\mathbf{N} + \mathbf{M}$ c $4\mathbf{L} + 3\mathbf{N}$
d $2\mathbf{L} + 5\mathbf{N} + \mathbf{M}$ e $3\mathbf{N} - 6\mathbf{M} - 5\mathbf{L}$

4 Given that

$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & -10 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}$, find:

a $\mathbf{C} - \mathbf{A}$ b $\mathbf{B} - \mathbf{D}$ c $\mathbf{C} - \mathbf{B}$ d $5\mathbf{A} - \mathbf{B}$ e $3\mathbf{B} - 2\mathbf{D}$

5 Given that

$\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix}$ and $\mathbf{Z} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 2 & 4 \\ 0 & 6 & 2 \\ 7 & 0 & 0 \end{bmatrix}$, find:

a $\mathbf{Z} - \mathbf{X}$ b $\mathbf{X} - \mathbf{Z}$ c $\mathbf{Y} - \mathbf{Z}$ d $4\mathbf{Z} - \mathbf{X}$ e $2\mathbf{Z} - 3\mathbf{X} + 2\mathbf{Y}$

6 Given that

$\mathbf{L} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 6 & -5 & 4 & -3 \\ -1 & 8 & -9 & 2 \\ 4 & 7 & -2 & 5 \end{bmatrix}$, find:

a $\mathbf{L} - \mathbf{M}$ b $\mathbf{N} - \mathbf{M}$ c $4\mathbf{L} - 3\mathbf{N}$
d $2\mathbf{L} + 5\mathbf{M} - \mathbf{N}$ e $3\mathbf{N} - 6\mathbf{M} - 5\mathbf{L}$

Reasoning and communication

7 A bank has three branches in Melbourne: City, Essendon and St Kilda. There are four types of accounts: Low Fee, Business, Savings and Student/Pensioner. The numbers of each type of account at the beginning of a year were as shown in the table below.

	Low Fee	Business	Savings	Stud/Pens
City	300	400	200	150
Essendon	400	220	300	200
St Kilda	250	150	400	150

The numbers of closures of accounts in that year were:

	Low Fee	Business	Savings	Stud/Pens
City	30	70	20	20
Essendon	20	30	40	30
St Kilda	40	10	50	10

The numbers of new accounts opened in that year were:

	Low Fee	Business	Savings	Stud/Pens
City	50	100	40	15
Essendon	15	50	30	10
St Kilda	30	25	20	40

Use 3 × 4 matrices to show:

a the number of accounts at the beginning of the year

b the number of closures in the year

c the number of accounts opened in the year

d an evaluated matrix expression for the number of accounts at the end of the year.

8 The results of the first five rounds, middle five rounds and last five rounds of a netball competition are shown by the following three matrices. Each row represents the results of a team. The first column is the number of games won, the second is drawn games and the third is games lost.

Newspix/George Salpigtidis

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

a Write a matrix expression for the total results.

b Evaluate your expression to obtain the final results.

c Which team (1st to 6th) is the top team?

8.04 MATRIX MULTIPLICATION

Matrices provide a useful way to write numerical information that is categorised simultaneously in two ways. The purchase of items from suppliers at different prices and the performances of bank branches with different types of accounts are two of the applications you have already seen. In many cases, the main categories are further divided into subcategories, leading to the use of related matrices.

A supplier of bulk meat makes up different packs so that hotels, hostels and caterers may order different mixes of meat to suit their requirements. The quantities, in kilograms, of different meats in each type of pack are shown in the table below.

Pack	Sausages	Mince	Chops	Stewing Steak	BBQ Steak	Prime Steak
BBQ	20	10	15	0	25	0
Basic	15	15	10	20	5	5
Quality	5	20	20	10	0	15
Deluxe	5	15	15	5	0	30

This can be written as the matrix:

$$\mathbf{B} = \begin{bmatrix} 20 & 10 & 15 & 0 & 25 & 0 \\ 15 & 15 & 10 & 20 & 5 & 5 \\ 5 & 20 & 20 & 10 & 0 & 15 \\ 5 & 15 & 15 & 5 & 0 & 30 \end{bmatrix}$$


Shutterstock.com/Kitch Bain

Over a period of 3 months, a caterer specialising in home parties orders the following packs.

Month	BBQ	Basic	Quality	Deluxe
July	2	3	1	0
August	3	2	2	1
September	4	1	1	3

This can be written as the matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 1 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

The two matrices can be combined to find the total quantity of each type of meat ordered each month. We need to combine the packs for each month with the quantities of meat in each pack. That is, we need to combine the matrices in the order

$$\mathbf{C} \times \mathbf{B} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 1 \\ 4 & 1 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 20 & 10 & 15 & 0 & 25 & 0 \\ 15 & 15 & 10 & 20 & 5 & 5 \\ 5 & 20 & 20 & 10 & 0 & 15 \\ 5 & 15 & 15 & 5 & 0 & 30 \end{bmatrix}$$

To obtain the quantity of sausages ordered in July, we work out:

$$2 \times 20 + 3 \times 15 + 1 \times 5 + 0 \times 5 = 90 \text{ kg}$$

To obtain the quantity of mince ordered in July, we work out:

$$2 \times 10 + 3 \times 15 + 1 \times 20 + 0 \times 15 = 85 \text{ kg}$$

For each type of meat in July, we multiply the packs ordered in July by the composition of each pack, and add the results.

Similarly, for each type of meat in August, we multiply the elements of the August row by the corresponding elements of the meat column.

Placing each month's meats in a row, and each type of meat in a column, we obtain:

$$\mathbf{A} = \begin{bmatrix} 90 & 85 & 80 & 70 & 65 & 30 \\ 105 & 115 & 120 & 65 & 85 & 70 \\ 115 & 120 & 135 & 45 & 105 & 110 \end{bmatrix}$$

The matrix gives the total amount of each meat each month, and could be written in table form as follows.

Month	Sausages	Mince	Chops	Stewing Steak	BBQ Steak	Prime Steak
July	90	85	80	70	65	30
August	105	115	120	65	85	70
September	115	120	135	45	105	110

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Although the operation we have performed on **C** and **B** involves both multiplication and addition, we call it **matrix multiplication**. We write either **A** = **CB** or **A** = **C** × **B**.

IMPORTANT

The **product** of two matrices **A** and **B** exists if and only if **A** has the same number of columns as **B** has rows. Matrices that fulfill this condition are called **conformable matrices**. If **A** is an $n \times m$ matrix (a_{ij}) and **B** is an $m \times p$ matrix (b_{ij}), then the product matrix **C** is an $n \times p$ matrix (c_{ij}) such that c_{ij} is the sum of the products of elements in the ith row of **A** and the corresponding elements of the jth column of **B**.

That is, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

Example 5

Find the product **AB** of matrices $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$.

Solution

Call the product **P** for convenience.

$$\mathbf{P} = \mathbf{AB} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$$

Multiply the first *row by* the first *column*.

$$p_{11} = 2 \times 1 + 3 \times 8 + (-1) \times 6 = 20$$

Multiply the second *row by* the first *column*.

$$p_{12} = 4 \times 1 + 2 \times 8 + 2 \times 6 = 32$$

Put the numbers in their correct places and write the product of **A** and **B**.

$$\mathbf{AB} = \begin{bmatrix} 20 \\ 32 \end{bmatrix}$$

Example 6

Find the product **FE** of the matrices $\mathbf{E} = \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 0 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} -1 & 0 & -2 & -3 \\ 1 & 2 & 2 & 1 \\ -3 & 4 & -4 & 0 \\ 2 & -5 & -1 & 3 \end{bmatrix}$.

Solution

F has to be first. Call the product **P** for convenience.

$$\mathbf{P} = \mathbf{FE} = \begin{bmatrix} -1 & 0 & -2 & -3 \\ 1 & 2 & 2 & 1 \\ -3 & 4 & -4 & 0 \\ 2 & -5 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 0 & 3 \\ 4 & 2 \end{bmatrix}$$

Multiply the first *row by* the first *column*.

$$p_{11} = (-1) \times 1 + 0 \times (-2) + (-2) \times 0 + (-3) \times 4 = -13$$

Multiply the first *row by* the second *column*.

$$p_{12} = (-1) \times 4 + 0 \times 1 + (-2) \times 3 + (-3) \times 2 = -16$$

Put these answers in place.

$$\mathbf{P} = \mathbf{FE} = \begin{bmatrix} -13 & 16 \end{bmatrix}$$

Now do the second row.

$$p_{21} = 1 \times 1 + 2 \times (-2) + 2 \times 0 + 1 \times 4 = 1$$

$$p_{22} = 1 \times 4 + 2 \times 1 + 2 \times 3 + 1 \times 2 = 14$$

Put the answers in place.

$$\mathbf{P} = \mathbf{FE} = \begin{bmatrix} -13 & 16 \\ 1 & 14 \end{bmatrix}$$

Continue with the other two rows.

$$p_{31} = (-3) \times 1 + 4 \times (-2) + (-4) \times 0 + 0 \times 4 = -11$$

$$p_{32} = (-3) \times 4 + 4 \times 1 + (-4) \times 3 + 0 \times 2 = -20$$

$$p_{41} = 2 \times 1 + (-5) \times (-2) + (-1) \times 0 + 3 \times 4 = 24$$

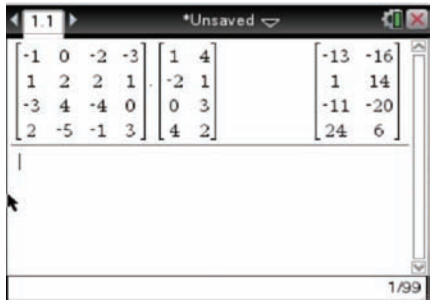
$$p_{42} = 2 \times 4 + (-5) \times 1 + (-1) \times 3 + 3 \times 2 = 6$$

Write the product of **F** and **E**.

$$\mathbf{FE} = \begin{bmatrix} -13 & -16 \\ 1 & 14 \\ -11 & -20 \\ 24 & 6 \end{bmatrix}$$

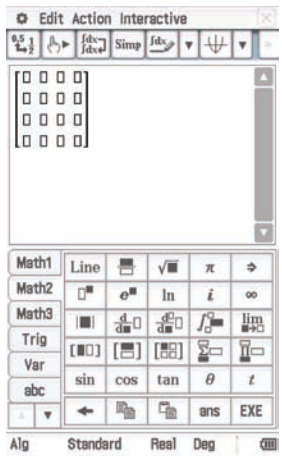
TI-Nspire CAS

From the menu, choose 7:Matrix & Vector, 1:Create and 1: Matrix. Select 4 rows and 4 columns for **F** and fill the matrix with elements. Now multiply × by **E**, creating a 4 × 2 matrix.

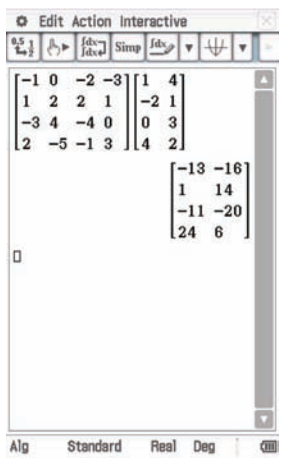


ClassPad

Matrices can be entered directly using the Math2 keyboard.
Press Keyboard then tap Math2.
Note the effect of the following keys.
[▮▯]: Starts a 1×2 matrix; adds a column.
[▤]: Starts a 2×1 matrix; adds a row.
[▦]: Starts a 2×2 matrix; adds a row and a column.
The easiest way to get a 4×4 matrix is to tap [▦] three times.



Enter the numbers in the matrix by tapping each square and typing the number. You can also use the arrows to move around the elements of the matrix.
To start the second matrix, tap [▮▯], then tap [▤] three times to add three more rows.
There is no need for a multiplication sign.
Enter the numbers and press EXE.



You could use Define() in Example **6** so that once you had set up **F** and **E**, they could be used again. You would only do this if you had a number of operations to perform with the same matrices.

To multiply matrices, the number of columns in the first matrix *must* equal the number of rows in the second matrix.

If **C** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then **CB** can be calculated and it is an $m \times p$ matrix. However, if we try to multiply **BC**, we find that the number of columns and rows are not equal as required and so the multiplication is not defined.

Thus, in general, matrix multiplication is not commutative, **AB** ≠ **BA**. However, it is associative so that **A**(**BC**) = (**AB**)**C**, although the general proof of associativity is quite difficult. For matrices of small sizes we can show that associativity is true, using the properties of real numbers.

Example 7

Prove that products of matrices are associative, where the first matrix is 1×2, the second is 2×2 and the third is 2×1.

Solution

State what has to be proved.	**RTP** $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 1×2, 2×2 and 2×1 respectively.
Set up general matrices for a proof.	**Proof** Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Start with the LHS.	$\text{LHS} = \mathbf{A}(\mathbf{BC})$ $= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} \right)$
Work out **BC**.	$= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} \\ b_{21}c_{11} + b_{22}c_{21} \end{bmatrix}$
Now multiply by **A**.	$= [a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21}]$
Rearrange.	$= [a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21}]$
Factorise.	$= [(a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21}]$
Express as a product.	$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Express the first matrix as a product.	$= \left(\begin{bmatrix} a_{11} & a_{11} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Write in symbols.	$= \mathbf{AB}(\mathbf{C}) = \text{RHS}$
Complete proof.	Since LHS = RHS, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ **QED**

EXERCISE 8.04 Matrix multiplication

Concepts and techniques

Multiplying matrices

1 Example 5 Calculate the following.

a $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$

e $\begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

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2 Example 6 Calculate the following.

a $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 4 & -2 \end{bmatrix}$ b $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 5 & -1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ c $\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ -1 & 2 \\ 4 & -3 \end{bmatrix}$

d $\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -5 \\ 1 & -2 \\ -4 & 3 \end{bmatrix}$ e $\begin{bmatrix} 1 & 5 & -3 \\ 2 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 4 \\ 4 & -3 & 2 \end{bmatrix}$

3 Calculate $\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}$ and comment on the result.

4 Calculate $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and comment on the result.

Reasoning and communication

5 Example 7 Prove that **A**(**B** + **C**) = **AB** + **AC** for 2 × 2 matrices **B** and **C** and a 1 × 2 matrix **A**. What is the name of this law?

6 Prove that matrix multiplication is associative for products of 2 × 2 matrices in general.

7 A kitchenware chain has four stores ordering stock from a central warehouse. The orders for cutlery sets for a catalogue special are as follows.

Setting	Store A	Store B	Store C	Store D
6 place	12	15	10	12
8 place	10	8	10	12
10 place	8	4	10	4

a Write the orders as a 4 × 3 matrix.
b The prices of the sets are \$117.80, \$155.00 and \$190.00 respectively. Write the prices as a column matrix.
c Multiply the matrices to obtain the invoice amounts for each store.

8 The costs of trucking small parcels between Brisbane, Mt Isa, Rockhampton, Sydney and Toowoomba are shown by the following matrix, where the rows give the costs from each place in alphabetical order.

$$\mathbf{C} = \begin{bmatrix} 0 & 10.80 & 9.80 & 6.40 & 5.00 \\ 10.80 & 0 & 7.20 & 16.40 & 15.40 \\ 9.80 & 7.20 & 0 & 16.20 & 14.40 \\ 6.40 & 16.40 & 16.20 & 0 & 11.00 \\ 5.00 & 15.40 & 14.40 & 11.00 & 0 \end{bmatrix}$$

a Why does the diagonal have zeros?
b The handling costs for parcel delivery are a flat \$4. Write a matrix **H** to give the handling costs and calculate the matrix **T** to give the total delivery costs for small parcels.
c Medium-sized parcels cost $1\frac{1}{2}$ times as much for trucking but the handling costs are the same. Write a matrix expression for the delivery costs for medium-sized parcels.
d Large parcels cost twice as much for handling and three times as much for trucking as small parcels. Write a matrix expression for the delivery costs of large parcels.

9 The results of the first five rounds, middle five rounds and last five rounds of a netball competition are shown in the following three matrices. Each row represents the results of a team. The first column is the number of games won, the second is the number of drawn games and the third is the number of games lost.

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

Team scores are calculated by allotting 3 points for a win, 1 point for a draw and 0 for a loss.

a Write the points as a column matrix **P**.

b Use matrix multiplication to find the team points for each group of five rounds.

c Calculate (**F** + **M** + **L**)**P** and **FP** + **MP** + **LP** and comment on your results.

8.05 IDENTITIES AND INVERSES

You need to understand multiplicative identity and inverse matrices to solve matrix equations.

IMPORTANT

The **identity matrix**, $\mathbf{I}_n$ is such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all $n \times n$ matrices **A**.

Since the identity and matrices must commute, an identity can exist only for square matrices.

Example 8

Show that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for 2×2 matrices.

Solution

Specifically state what has to be proved.

RTP

$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A} = \mathbf{A}$ for all 2×2 matrices **A**.

Choose a general matrix.

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Left multiply.

$$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \times 1 + a_{12} \times 0 & a_{11} \times 0 + a_{12} \times 1 \\ a_{21} \times 1 + a_{22} \times 0 & a_{21} \times 0 + a_{22} \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}$$

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Right multiply.	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathbf{A} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$
	$= \begin{bmatrix}1\times a_{11}1+0\times a_{12} & 0\times a_{11}+1\times a_{12}\\1\times a_{21}+0\times a_{22} & 0\times a_{21}+1\times a_{22}\end{bmatrix}$
	$= \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} = \mathbf{A}$
Write the conclusion.	Thus $\mathbf{A}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathbf{A} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$. QED

The function δ_{ij} is called the **Kronecker delta** function and is used in many areas of mathematics.

In the real numbers, 1 is the only number that can be squared to give itself and 0 is the only number that can be squared to give 0. This is not true for matrices.

Since there is a multiplicative identity, it is possible that an inverse will also exist.

IMPORTANT

The **identity matrix**, $\mathbf{I}_n$ has elements of 1 along the **leading diagonal** and zeros for the other elements.
$\mathbf{I}_n = (\delta_{ij})$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

IMPORTANT

If $\mathbf{A}$ is a square matrix, then its **inverse matrix** is denoted by $\mathbf{A}^{-1}$.

$\mathbf{A}^{-1}$ is the matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the **identity matrix**.

A matrix that has an inverse is called **invertible** or **non-singular**.

A matrix for which no inverse exists is said to be **singular**.

A matrix that squares to give itself ($\mathbf{A}^2 = \mathbf{A}$) is called **idempotent**.

A matrix that squares to give the zero matrix ($\mathbf{A}^2 = \mathbf{0}$) is called **nilpotent**.

Example 9

Find the inverse matrix for $\mathbf{X} = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$.

Solution

Write a general matrix for the inverse.

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{I}$.

Work out the matrix product.

$$\mathbf{AX} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a \times (-4) + b \times 1 & a \times 2 + b \times 0 \\ c \times (-4) + d \times 1 & c \times 2 + d \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} b - 4a & 2a \\ d - 4c & 2c \end{bmatrix}$$

Now write the equality.

$$\begin{bmatrix} b - 4a & 2a \\ d - 4c & 2c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Write the equation from the equality.

$b - 4a = 1$, $2a = 0$, $d - 4c = 0$ and $2c = 1$

Solve for a and c.

$a = 0$ and $c = \frac{1}{2}$

Find b and d.

$b = 1$ and $d = 2$

Write **A**.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$$

Check the value of **AX**.

$$\mathbf{AX} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \times (-4) + 1 \times 1 & 0 \times 2 + 1 \times 0 \\ \frac{1}{2} \times (-4) + 2 \times 1 & \frac{1}{2} \times 2 + 2 \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now work out **XA**.

$$\mathbf{XA} = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -4 \times 0 + 2 \times \frac{1}{2} & -4 \times 1 + 2 \times 2 \\ 1 \times 0 + 0 \times \frac{1}{2} & 1 \times 1 + 0 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Write the result.

$\mathbf{AX} = \mathbf{XA} = \mathbf{I}$, so $\mathbf{X}^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$

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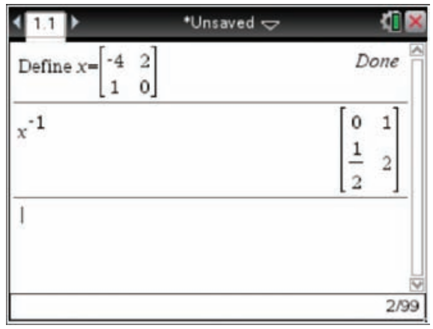
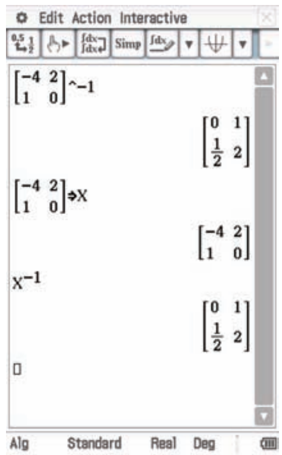
TI-Nspire CAS

Define the matrix **X** (see page 288).
Now use [^] to find the inverse of the matrix (–1).



ClassPad

Find the inverse of the matrix (–1) using either [^] or tapping [□■] in the [Math1] or [Math2] keyboard.
It doesn't have to be defined as **X**.



IMPORTANT

For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse is the matrix $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The quantity $ad - bc$ is called the **determinant**, shown symbolically as det **A**, |**A**| or *A*.

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, which is a real number.

The inverse of a matrix can be written as $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided det $\mathbf{A} \neq 0$.

A **singular** matrix is one with no inverse.

A matrix is singular if its determinant is zero.

Example 10

For the matrix $\mathbf{M} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, find each of the following using the formulas for 2×2 matrices.

a det **M**

b $\mathbf{M}^{-1}$

Solution

a Write the formula. det $\mathbf{M} = ad - bc$

Substitute values and simplify. $= 6 \times 3 - (-1) \times 2 = 20$

Write the answer. $= 20$

b Write the formula.

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Substitute values and simplify.

$$= \frac{1}{20}\begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$$

Write the answer.

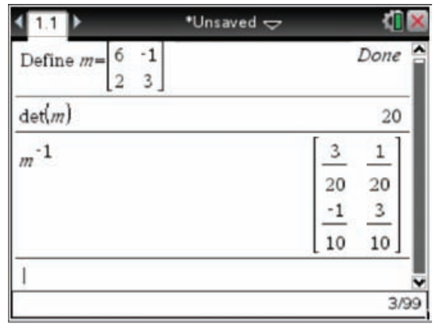
$$= \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

TI-Nspire CAS

Enter the matrix as **M**.

Use [menu] 7: Matrices & Vectors and 3: Determinant to find the determinant.

Find $\mathbf{M}^{-1}$.

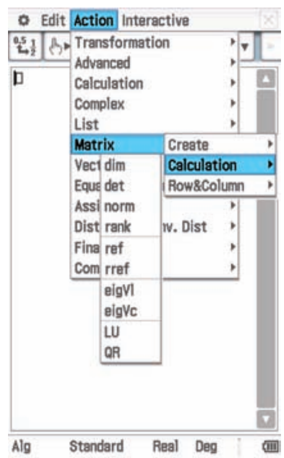


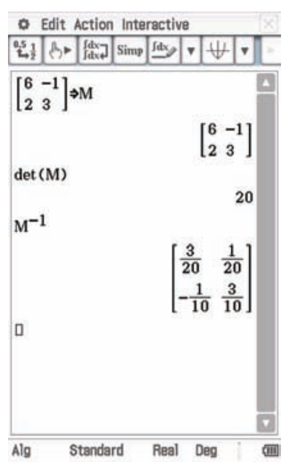
ClassPad

Define the matrix as **M**.

det can be typed in or found in the menu obtained by tapping **Action** then **Matrix** then **Calculation**.

Find $\mathbf{M}^{-1}$ or M^−1.





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IMPORTANT

Laws for matrix multiplication

The following laws hold for any matrices **A, B** and **C** and scalars *r* and *s*.

1. Matrix multiplication is associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. Matrix multiplication is *not commutative*: in general $\mathbf{AB} \neq \mathbf{BA}$
3. The multiplicative identity **I** is a square matrix with 1s in the leading diagonal and 0 everywhere else. For $n \times n$ matrices, $\mathbf{I}_n = (\delta_{ij})$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.
4. Matrices that have a multiplicative inverse are said to be **invertible** or **non-singular** matrices. Matrices without multiplicative inverses are called **singular** matrices.
5. Cancellation laws hold for invertible matrices.
   If **A** is invertible, then $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$ and $\mathbf{BA} = \mathbf{CA} \Rightarrow \mathbf{B} = \mathbf{C}$
6. Distributive laws hold for matrix multiplication over matrix addition.
   Left distributive law: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
   Right distributive law: $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
7. Matrix multiplication over addition: $\mathbf{A}(r + s) = r\mathbf{A} + r\mathbf{B}$
8. The associative law holds for matrix multiplication and scalar multiplication:
   $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B}$

There are left and right distributive laws because matrix multiplication is not commutative.

EXERCISE 8.05 Identities and inverses

Concepts and techniques

WS Inverse and determinant of a matrix

1. Example 9 Find the inverses of the following matrices using the method of Example **9**.

   a $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ **b** $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ **c** $\mathbf{A} = \begin{bmatrix} 3 & -3 \\ -1 & 0 \end{bmatrix}$

2. Find the inverses of the following matrices.

   a $\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$ **b** $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & \frac{1}{3} \end{bmatrix}$ **c** $\mathbf{A} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$

3. State whether each of the following is nilpotent.

   a $\begin{bmatrix} -4 & 2 \\ -8 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix}$

4. Which of the following matrices are idempotent?

   a $\begin{bmatrix} 5 & 4 \\ -5 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & -4 \\ 18 & -8 \end{bmatrix}$

5. Example 10 Use the formula to find the determinants of the following matrices.

   a $\begin{bmatrix} -12 & 2 \\ 1 & -3 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 6 \\ -6 & 11 \end{bmatrix}$ **c** $\begin{bmatrix} -7 & 3 \\ -3 & 3 \end{bmatrix}$

   d $\begin{bmatrix} -15 & 14 \\ 3 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 0 \\ -11 & 7 \end{bmatrix}$

6 Use your CAS calculator to find the determinant of each of the following.

a $\begin{bmatrix} 7 & -1 \\ -10 & 9 \end{bmatrix}$ b $\begin{bmatrix} 13 & 6 \\ 1 & -13 \end{bmatrix}$ c $\begin{bmatrix} -14 & -15 \\ 11 & 0 \end{bmatrix}$

d $\begin{bmatrix} 4 & -8 \\ 11 & 6 \end{bmatrix}$ e $\begin{bmatrix} -1 & 4 \\ -14 & 13 \end{bmatrix}$

7 Use the formula to find the inverses of the following, if they exist.

a $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ b $\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ c $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

d $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ e $\begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}$

8 Find the inverse of $\mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}$ using your CAS calculator.

Reasoning and communication

9 Find a matrix of order 2 that is both nilpotent and idempotent.

10 Given that $\mathbf{X} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} -5 & -3 \\ 4 & 3 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that:

a $\mathbf{XM} = \mathbf{X}$ but $\mathbf{MX} \neq \mathbf{X}$ b $\mathbf{NX} = \mathbf{X}$ but $\mathbf{XN} \neq \mathbf{X}$ c $\mathbf{XI} = \mathbf{X}$ and $\mathbf{IX} = \mathbf{X}$

11 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (assume $ad - bc \neq 0$).

a Show that $\mathbf{IA} = \mathbf{A}$. b Show that $\mathbf{AI} = \mathbf{A}$. c Show that $\mathbf{A} \times \mathbf{0} = \mathbf{0}$.

d Show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. e Show that $\mathbf{AA}^{-1} = \mathbf{I}$.

12 Prove that the inverse of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

13 A pizza shop makes pizzas in three sizes: small, medium and large. In one particular week the sales were recorded for each day and each size of pizza. This data is shown in the table.

	Number of pizzas sold						
Size	**Sun**	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**
Small	25	20	50	25	30	50	50
Medium	35	35	80	35	40	70	75
Large	45	40	100	40	40	90	85

a Write the information in the table as a 3×7 matrix, **P**.

b Find a matrix, **A**, such that when **P** is multiplied by **A**, the resulting matrix will give the number of each pizza size sold that week.

c Write the matrix **PA**.

d Find a matrix, **B**, such that when **B** is multiplied by **P**, the resulting matrix will give the number of pizzas sold on each day of the week.

e Write the matrix **BP**.

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The cost of a small pizza is \$7.50, a medium pizza is \$11.00 and a large pizza is \$15.00.

f Write a matrix, **C**, that represents the cost of the different-sized pizzas.

g Use an appropriate matrix method to find the income from each type of pizza sold that week.

8.06 MATRIX EQUATIONS

You can manipulate matrix expressions and equations in almost the same way as ordinary algebraic expressions and equations. The two exceptions are that matrix multiplication is not generally commutative, and you cannot divide matrices. Where you would normally divide, you must multiply by the inverse, provided that it exists.

You will often find that it is necessary to insert the identity matrix into an equation or expression to factorise it.

While it is correct to write $ab - a = a(b - 1)$ for real numbers, $\mathbf{AB} - \mathbf{A} \neq \mathbf{A}(\mathbf{B} - 1)$ because subtracting the number 1 from the matrix **B** does not make sense.

You must write $\mathbf{A} = \mathbf{AI}$ so that $\mathbf{AB} - \mathbf{AI} = \mathbf{A}(\mathbf{B} - \mathbf{I})$.

Notice that it would not be correct to use $\mathbf{A} = \mathbf{IA}$, because then **A** would not be on the left side as it is in **AB**.

Example 11

Factorise the following matrix expressions where possible.

a $\mathbf{AB} + \mathbf{B}$ **b** $\mathbf{BA} - \mathbf{B}$

c $\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I}$ **d** $\mathbf{X}^2 - 5\mathbf{XY} - 6\mathbf{Y}^2$

Solution

a	Write the matrix expression.	$\mathbf{AB} + \mathbf{B}$
	Insert the identity on the left to match **A**.	$= \mathbf{AB} + \mathbf{IB}$
	Use the right distributive law.	$= (\mathbf{A} + \mathbf{I})\mathbf{B}$
b	Write the matrix expression.	$\mathbf{BA} - \mathbf{B}$
	Insert the identity on the right to match **A**.	$= \mathbf{BA} - \mathbf{BI}$
	Use the left distributive law.	$= \mathbf{B}(\mathbf{A} - \mathbf{I})$
c	Use the decomposition method as you would for $x^2 - 3x - 10$.	$-10 = -5 \times (+2)$ $-3 = -5 + (+2)$
	Write the matrix expression.	$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I}$
	Break up the terms to produce the common factor $(\mathbf{A} - 5\mathbf{I})$.	$= \mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{A} - 10\mathbf{I}$
	Insert the identity on the left and right.	$= \mathbf{A} \times \mathbf{A} - 5\mathbf{I} \times \mathbf{A} + \mathbf{A} \times 2\mathbf{I} - 5\mathbf{I} \times 2\mathbf{I}$
	Use the right distributive law twice.	$= (\mathbf{A} - 5\mathbf{I})\mathbf{A} + (\mathbf{A} - 5\mathbf{I})2\mathbf{I}$
	Use the left distributive law.	$= (\mathbf{A} - 5\mathbf{I})(\mathbf{A} + 2\mathbf{I})$
d	Use the decomposition method as you would for $x^2 - 5xy - 6y^2$.	$-6 = -6 \times (+1)$ $-5 = -6 + (+1)$
	Write the matrix expression.	$\mathbf{X}^2 - 5\mathbf{XY} - 6\mathbf{Y}^2$
	Break up the terms to produce $\mathbf{X} - 6\mathbf{Y}$.	$= \mathbf{X}^2 - 6\mathbf{XY} + \mathbf{XY} - 6\mathbf{Y}^2$

Write each pair of terms so it can be factorised..	$= \mathbf{X}^2 - \mathbf{X} \times 6\mathbf{Y} + 1\mathbf{XY} - 6\mathbf{Y}^2$
Apply the left and right distributive laws. No further factorisation can be done because the $\mathbf{X} - 6\mathbf{Y}$ terms are on different sides.	$= \mathbf{X}(\mathbf{X} - 6\mathbf{Y}) + (\mathbf{X} - 6\mathbf{Y})\mathbf{Y}$

As you can see in Example **11** part **d**, you can factorise $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$ or $\mathbf{BA} + \mathbf{CA} = (\mathbf{B} + \mathbf{C})\,\mathbf{A}$, but they are not the same and you cannot factorise $\mathbf{AB} + \mathbf{CA}$.

Example 12

Solve the following equations for **X** and **Y**.

a $\mathbf{X}\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$

Solution

a Write the matrix equation.	$\mathbf{X}\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$
To find **X**, the inverse of $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ must exist.	For $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$, $\det \mathbf{A} = -1$
Since $\det \mathbf{A} \neq 0$, the inverse exists.	$\mathbf{A} = \frac{1}{-1}\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$
Multiply the left side of both sides of the equation by $\mathbf{A}^{-1}$. Let $\mathbf{B} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.	$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$
Simplify $\mathbf{AA}^{-1}$.	$\mathbf{XI} = \mathbf{BA}^{-1}$
Simplify **XI**.	$\mathbf{X} = \mathbf{BA}^{-1}$
Substitute the matrices.	$\mathbf{X} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$
Multiply.	$= \begin{bmatrix} -28 & 22 \\ 19 & -14 \end{bmatrix}$
b Write the matrix equation.	$\mathbf{AY} = \mathbf{B}$
Multiply the right side of both sides of the equation by $\mathbf{A}^{-1}$.	$\mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{B}$
Simplify $\mathbf{A}^{-1}\mathbf{A}$.	$\mathbf{IY} = \mathbf{A}^{-1}\mathbf{B}$
Simplify **IY**.	$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$
Substitute the matrices.	$\mathbf{Y} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$
Multiply.	$= \begin{bmatrix} -16 & 34 \\ 13 & -26 \end{bmatrix}$

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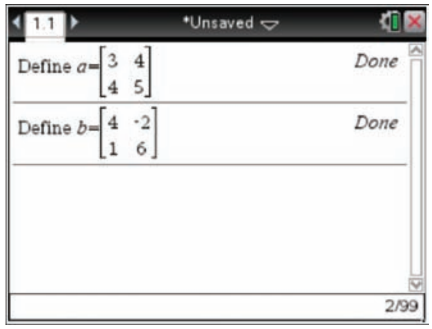
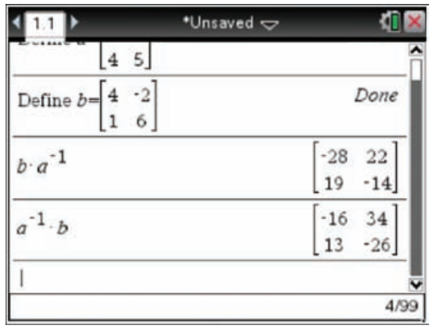
TI-Nspire CAS

Enter the matrices **A** and **B**.



Calculate $\mathbf{BA}^{-1}$ and $\mathbf{A}^{-1}\mathbf{B}$.



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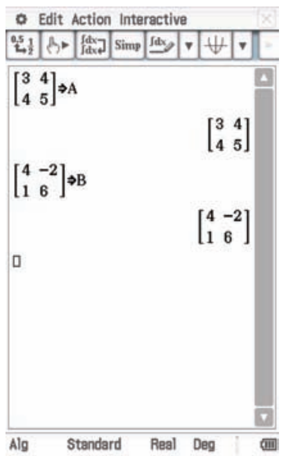
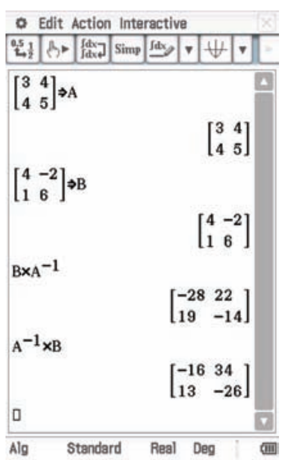
Enter the matrices **A** and **B**.



Calculate $\mathbf{B} \times \mathbf{A}^{-1}$ and $\mathbf{A}^{-1} \times \mathbf{B}$.
The multiplication sign is essential in this case.



The order of calculations is very important because matrix multiplication is not commutative ($\mathbf{AB} \neq \mathbf{BA}$).

Example 13

Solve the following equation using your CAS calculator as needed.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}$$

Solution

Write the equation.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}$$

Take **X** terms to the LHS and the others to the RHS.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Reorder the second term.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - \mathbf{X}\times 2\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Factorise.

$$\mathbf{X}\left(\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 2\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Write in simplified form.

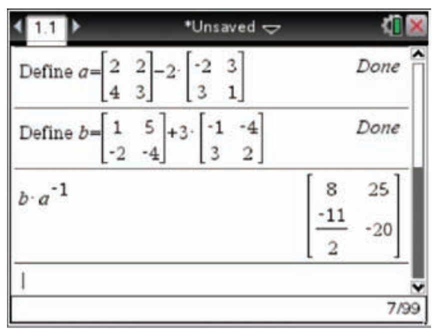
$\mathbf{XA} = \mathbf{B}$

Perform the necessary operation.

$\mathbf{X} = \mathbf{BA}^{-1}$

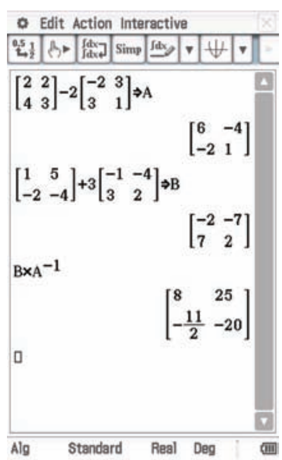
TI-Nspire CAS

Perform the matrix operations and call the results **A** and **B**.
Calculate $\mathbf{BA}^{-1}$.



ClassPad

Perform the matrix operations and call the results **A** and **B**.
Calculate $\mathbf{BA}^{-1}$.



Write the answer.

$$\mathbf{X} = \begin{bmatrix}8 & 25\\-5\frac{1}{2} & -20\end{bmatrix}$$

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EXERCISE 8.06 Matrix equations

Concepts and techniques

1 Example 11 Factorise each of the following matrix expressions, if possible.

a $\mathbf{M}^2 + 4\mathbf{M}$ b $7\mathbf{GX} - \mathbf{X}$ c $\mathbf{AB} - 3\mathbf{A}$

d $\mathbf{H}^2\mathbf{G} + 4\mathbf{HG}^2$ e $\mathbf{X}^2 - 2\mathbf{XY} + \mathbf{Y}^2$

2 Example 12 Solve the following.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 & 3 \\ -1 & -4 \end{bmatrix}$ b $\mathbf{X}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}$ c $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 17 \\ 9 \end{bmatrix}$

d $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ e $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 & -2 \\ 7 & -5 \end{bmatrix}$

3 Solve the following.

a $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 & 4 \\ -6 & -8 \end{bmatrix}$ b $\mathbf{X}\begin{bmatrix} -4 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 10 & -4 \end{bmatrix}$ c $\begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d $\mathbf{X}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -5 & 3 \end{bmatrix}$ e $\begin{bmatrix} -5 & 2 \\ -10 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 & -2 & 5 \\ 3 & 0 & -3 \end{bmatrix}$

Reasoning and communication

4 Example 13 Solve the following, using your CAS calculator as needed.

a $2\mathbf{X} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{X} - \begin{bmatrix} 2 & 12 \\ 20 & 12 \end{bmatrix} = \begin{bmatrix} -14 & -6 \\ 22 & -3 \end{bmatrix}$ b $\begin{bmatrix} 11 & 6 \\ 3 & 5 \end{bmatrix}\mathbf{X} - 3\mathbf{X} + \begin{bmatrix} 4 & 6 \\ 4 & 3 \end{bmatrix} = 4\mathbf{I}$

c $2\mathbf{X} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{X} - 4\mathbf{X} = \mathbf{X} + \begin{bmatrix} -14 & -6 \\ 22 & -3 \end{bmatrix}$ d $5\mathbf{X} - \mathbf{X}\begin{bmatrix} -1 & 5 \\ 2 & 1 \end{bmatrix} + 7\mathbf{I} = \mathbf{0}$

e $\mathbf{X}\begin{bmatrix} 8 & -2 & 6 \\ -5 & 5 & 4 \\ -4 & 2 & 5 \end{bmatrix} - 2\mathbf{X} + 3\mathbf{I} = \begin{bmatrix} -9 & 3 & 1 \\ 2 & 6 & 2 \\ 3 & 4 & -3 \end{bmatrix}$

8 CHAPTER SUMMARY
MATRIX ARITHMETIC

- A **matrix** (plural = matrices) is a rectangular array of **elements** shown in rows and columns enclosed by brackets, such as $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$. The **size** (or **dimension** or **order**) of a matrix is stated with the number of rows first, so **A** is a 2×3 matrix.
- A **square** matrix has the same number of rows and columns, while a row matrix has only 1 row and a column matrix has only 1 column.
- A general matrix may be written as $\mathbf{C} = c_{ij}$ or $(c_{ij})_{m\times n}$ if the size needs to be shown.
- **Matrix addition** is only defined for matrices of the same size. For the $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ such that $c_{ij} = a_{ij} + b_{ij}$ for all i, j; i.e. $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.
- Multiplying a matrix by a constant (c) produces a **scalar multiple** of the matrix, so $c\mathbf{A} = (ca_{ij})$. The product $(-1)\mathbf{A}$ is normally written as $-\mathbf{A}$, so matrix subtraction can be shown as $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = \mathbf{A} + (-\mathbf{B})$.
- **A** and **B** are **equal** if and only if they are the same size and $a_{ij} = b_{ij}$, for all i, j.
- The laws for matrix addition and scalar multiplication are:

  Associativity:
  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, $(rs)\mathbf{A} = r(s\mathbf{A})$

  Commutivity: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

  Additive identity: $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

  Additive inverse: $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$, where $\mathbf{A} = (a_{ij})$ and $-\mathbf{A} = (-a_{ij})$

  Cancellation: $r\mathbf{A} = r\mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{B}$ $(r \neq 0)$,
  $r\mathbf{A} = s\mathbf{A} \Leftrightarrow r = s$ $(\mathbf{A} \neq \mathbf{0})$

  Distributivity: $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$,
  $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$

- The product of two matrices **A** and **B** exists if and only **A** and **B** are **conformable**. This means that **A** has the same number of columns as **B** has rows. If **A** is an $n \times m$ matrix (a_{ij}) and **B** is an $m \times p$ matrix (b_{ij}) and $\mathbf{AB} = \mathbf{C}$, then **C** is an $n \times p$ matrix (c_{ij}) such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$.
- An **idempotent matrix A** is such that $\mathbf{A}^2 = \mathbf{A}$. A **nilpotent matrix B** of order n is such that $\mathbf{B}^2 = \mathbf{0}_n$, where $\mathbf{0}_n$ is the null matrix of size $n \times n$.
- **Identity matrices** (**I**) are diagonal matrices with 1s in the diagonal and 0s everywhere else. $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.
- The multiplicative inverse of a 2×2 matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The quantity $ad - bc$ is a number called the **determinant** of the matrix.
- Matrix multiplication is not commutative.

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- The laws of matrix multiplication are:

  Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

  Identity: $\mathbf{I}$ is a square matrix with 1s in the leading diagonal and 0 everywhere else. For $n \times n$ matrices $\mathbf{I}_n = (\delta_{ij})$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

  Inverse: matrices that have a multiplicative inverse are called **invertible** or **non-singular** matrices. Matrices without multiplicative inverses are called **singular** matrices.

  Cancellation: If $\mathbf{A}$ is invertible, then $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$ and $\mathbf{BA} = \mathbf{CA} \Rightarrow \mathbf{B} = \mathbf{C}$

  Left distributive law: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

  Right distributive law: $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$

  Distributivity over addition: $\mathbf{A}(r + s) = r\mathbf{A} + s\mathbf{A}$

  Associativity with scalar multiplication: $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B}$

- Matrix algebra differs from the algebra of real numbers because matrix multiplication is not commutative and there is no division. It is often necessary to introduce the identity to complete factorisations in the solution of matrix equations.

8 CHAPTER REVIEW
MATRIX ARITHMETIC

Multiple choice

1 Example 1 What is the size of the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ -1 & 3 \end{bmatrix}$?

A 2×3 B 3×2 C 2×2 D 1×3 E 6

2 Example 2 Given that $\mathbf{X} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$, which matrix represents $-2\mathbf{X}$?

A $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ B $\begin{bmatrix} 0 & -2 & 4 \\ -2 & 0 & -2 \\ 4 & -2 & 0 \end{bmatrix}$ C $\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$

D $\begin{bmatrix} 0 & 2 & -2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}$ E $\begin{bmatrix} 0 & 2 & -4 \\ 2 & 0 & 2 \\ -4 & 2 & 0 \end{bmatrix}$

3 Example 3 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 \end{bmatrix}$, choose the correct answer for $3\mathbf{A} + 2\mathbf{B}$.

A It cannot be done. B $\begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$ C $\begin{bmatrix} 6 & 3 \end{bmatrix}$

D $\begin{bmatrix} 15 & 4 \\ 3 & 0 \end{bmatrix}$ E $\begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$

4 Example 4 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$, then $3\mathbf{A} - 2\mathbf{B}$ is equal to:

A $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ B $\begin{bmatrix} 8 & -3 \\ 3 & 2 \end{bmatrix}$ C $\begin{bmatrix} 7 & 0 \\ 3 & 4 \end{bmatrix}$

D $\begin{bmatrix} 11 & -12 \\ 3 & 4 \end{bmatrix}$ E $\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$

5 Example 10 What is the inverse of $\begin{bmatrix} 6 & -11 \\ -2 & 4 \end{bmatrix}$?

A $\begin{bmatrix} 3 & -1 \\ -5\frac{1}{2} & 2 \end{bmatrix}$ B $\begin{bmatrix} 2 & 5\frac{1}{2} \\ 1 & 3 \end{bmatrix}$ C $\begin{bmatrix} 2 & 1 \\ 5\frac{1}{2} & 3 \end{bmatrix}$

D $\begin{bmatrix} \frac{2}{23} & -\frac{11}{46} \\ -\frac{1}{23} & \frac{3}{23} \end{bmatrix}$ E $\begin{bmatrix} -\frac{2}{23} & \frac{11}{46} \\ \frac{1}{23} & -\frac{3}{23} \end{bmatrix}$

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Short answer

6 Example 5 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$, find **BA**.

7 Example 6 If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, find **AB**.

8 Example 9 Find the inverse of $\begin{bmatrix} -6 & 2 \\ -5 & 2 \end{bmatrix}$ from the definition (without using a formula).

9 Example 10 For each of the following matrices, find the determinant and inverse, if they exist.

a $\begin{bmatrix} 3 & 3 \\ 8 & 9 \end{bmatrix}$

b $\begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix}$

10 Example 11 Factorise $\mathbf{P}^2\mathbf{Q} + \mathbf{P}\mathbf{Q}^2$ using matrix algebra, if possible.

11 Example 12 Solve the matrix equation $\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix}\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

12 Example 13 Solve the matrix equation $\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix}\mathbf{C} + 3\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} = 2\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, using your CAS calculator as needed.

Application

13 Show that matrices **A**, **B** and **C**, each with dimensions of 2×2, are associative under addition, i.e. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

14 A smallgoods manufacturer produces three types of salami – Italian, Hungarian and Austrian – in mild and hot versions.
The prices in dollars of each type are shown in the table below.

Type of salami	Italian	Hungarian	Austrian
Mild	12.50	15.00	17.50
Hot	13.20	16.40	18.00

The orders from three delicatessens are as follows.

Orders	MI	HI	MH	HH	MA	HA
George's	3	1	2	3	1	0
Frank's	2	2	3	1	3	2
Maria's	4	5	0	1	0	2

Practice quiz

a Write the prices as a 1×6 matrix.
b Write the orders as a 6×3 matrix.
c Use matrix multiplication to find the price of each deli's order.

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